Complexity of decision problems on TRATGs

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2017-05-25
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Totally rigid acyclic tree grammars

Complexity

Cut-reduction
Terms, not words.

- Start symbol: \( A \)
- Nonterminals: \( A, B, C, D, \ldots \)
- (Acyclic) productions: \( B \rightarrow t[C, D, \ldots] \)

Rigid derivations: \( A[A\{t_1\}][B\{t_2\}][C\{t_3\}] \ldots \)

Language \( L(G) \) consists of all derivable terms
A \rightarrow f(B, B) \mid g(B, B)
B \rightarrow c \mid d
A \rightarrow f(B, B) | g(B, B)
B \rightarrow c | d

L(G) = \{ f(c, c), f(d, d), g(c, c), g(d, d) \}
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Complexity

Cut-reduction
Problem (Membership)
Given a TRATG $G$ and a term $t$, is $t \in L(G)$?
Membership

Problem (Membership)
Given a TRATG $G$ and a term $t$, is $t \in L(G)$?

Claim: Membership is NP-complete.

- Derivations of $t$ are polynomial in the size of $t$ and $G$.

  $A, A[A\backslash s_1], A[A\backslash s_1][B\backslash s_2], \ldots$

  Can check in polynomial time whether such a sequence of terms is a derivation of $t$ in $G$.

- Hardness: next slide.
The TRATG $\text{Sat}_{n,m}$ generates the satisfiable 3-CNFs with $n$ clauses and $m$ variables:

$$A \rightarrow \text{and}(\text{Clause}_1, \ldots, \text{Clause}_n)$$

$$\text{Clause}_i \rightarrow \text{or}(\text{True}_i, \text{Any}_{i,1}, \text{Any}_{i,2})$$

$$\text{Clause}_i \rightarrow \text{or}(\text{Any}_{i,1}, \text{True}_i, \text{Any}_{i,2})$$

$$\text{Clause}_i \rightarrow \text{or}(\text{Any}_{i,1}, \text{Any}_{i,2}, \text{True}_i)$$

$$\text{Any}_{i,k} \rightarrow x_1 | \neg(x_1) | \cdots | x_m | \neg(x_m) | \text{false} | \text{true}$$

$$\text{True}_i \rightarrow \text{Value}_1 | \cdots | \text{Value}_m | \text{true}$$

$$\text{Value}_j \rightarrow x_j | \neg(x_j)$$
Problem (Containment)

Given TRATGs $G_1$ and $G_2$, is $L(G_1) \subseteq L(G_2)$?
Containment

**Problem (Containment)**
*Given TRATGs $G_1$ and $G_2$, is $L(G_1) \subseteq L(G_2)$?*

Claim: $\Pi_2^P$-complete

- In $\Pi_2^P$: for every sequence of terms check if it is a derivation of a term $t$ in $G_1$, and then if $t \in L(G_2)$. 
Containment ($\Pi_2^P$-hardness)

- Determining the truth of the quantified Boolean formula 
  $\forall y_1 \ldots \forall y_k \exists x_1 \ldots \exists x_m f$ is $\Pi_2^P$-complete.
- Let $f$ be in 3-CNF with $n$ clauses.

- Is $f\sigma$ satisfiable for any $\sigma: \{y_1, \ldots, y_k\} \rightarrow \{\text{true, false}\}$?
- Is $\{f\sigma : \sigma: \{y_1, \ldots, y_k\} \rightarrow \{\text{true, false}\}\} \subseteq L(\text{Sat}_{n,m})$?
- Left side is generated by a TRATG:

  $A \rightarrow f[y_1 \backslash Y_1, \ldots, y_k \backslash Y_k]$

  $Y_j \rightarrow \text{true} \mid \text{false}$
Problem (Disjointness)
Given TRATGs $G_1$ and $G_2$, is $L(G_1) \cap L(G_2) = \emptyset$?

Problem (Equivalence)
Given TRATGs $G_1$ and $G_2$, is $L(G_1) = L(G_2)$?

$\Rightarrow$ Disjointness is coNP-complete (via Membership)

$\Rightarrow$ Equivalence is $\Pi_2^P$-complete (via Containment)
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Complexity

Cut-reduction
Proofs with $\Pi_1$-cuts

**Definition (simple proof)**
We call a proof $\pi$ in LK simple iff:

- The end-sequent is prenex $\Sigma_1$
- Cuts have at most a single quantifier, which is prenex
- Quantified cuts are immediately followed by a strong quantifier rule
We assign to every simple proof $\pi$ a TRATG $G(\pi)$. $L(G(\pi))$ contains the formulas in a Herbrand sequent of $\pi$

- Nonterminals: eigenvariables from cuts + start symbol $A$
- Productions $x \rightarrow t$ for weak quantifier inferences on cut formulas:

  $\vdash \varphi(x)$ \hspace{1cm} $\varphi(t) \vdash$ \hspace{1cm} $\forall$-l

  $\vdash \forall x \varphi(x)$ \hspace{1cm} $\forall$-r \hspace{1cm} $\vdash \forall x \varphi(x)$ \hspace{1cm} $\vdash \forall x \varphi(x)$ \hspace{1cm} $\vdash$ \hspace{1cm} $\vdash$ \hspace{1cm} $\vdash$

- Productions $A \rightarrow \varphi(t)$ for instances of formulas end-sequent.
Theorem ([Hetzl and Straßburger 2012])

- For every Gentzen cut-reduction sequence $\pi \leadsto \pi'$, we have $L(G(\pi)) \supseteq L(G(\pi'))$.
- If we did not perform grade reduction on weakenings, then $L(G(\pi)) = L(G(\pi'))$.

Let $\text{ne}$ be the non-erasing Gentzen cut-reduction relation, i.e. where we do not reduce weakenings.

We can directly extract tautological Herbrand sequents from $\text{ne}$-NFs.

$\Rightarrow f \in H(\pi^*)$ iff $f \in L(G(\pi))$ (for any $\text{ne}$-NF $\pi^*$)
Corresponding problems for simple proofs

Problem (H-membership)
Let $\pi$ be a simple proof, and $f$ a formula. Is there a $\neq\rightsquigarrow$-NF $\pi \neq\rightsquigarrow \pi^*$ such that $f \in H(\pi^*)$?

Problem (H-containment)
Let $\pi_1, \pi_2$ be simple proofs. Are there $\neq\rightsquigarrow$-NFs $\pi_i \neq\rightsquigarrow \pi_i^*$ such that $H(\pi_1^*) \subseteq H(\pi_2^*)$?

Problem (H-disjointness)
Let $\pi_1, \pi_2$ be simple proofs. Are there $\neq\rightsquigarrow$-NFs $\pi_i \neq\rightsquigarrow \pi_i^*$ such that $H(\pi_1^*) \cap H(\pi_2^*) = \emptyset$?

Problem (H-equivalence)
Let $\pi_1, \pi_2$ be simple proofs. Are there $\neq\rightsquigarrow$-NFs $\pi_i \neq\rightsquigarrow \pi_i^*$, such that $H(\pi_1^*) = H(\pi_2^*)$?
Lemma
There is a formula $\varphi(x)$ such that we can assign to every grammar $G$ a simple proof $\pi_G$ satisfying $H(\pi_G^*) = \varphi[L(G)]$ for any $\text{ne} \leadsto \text{-NF}$ $\pi_G^*$. 
Lemma

There is a formula $\varphi(x)$ such that we can assign to every grammar $G$ a simple proof $\pi_G$ satisfying $H(\pi^*_G) = \varphi[L(G)]$ for any ne-NF $\pi^*_G$.

Set $\varphi(x) := L(x) \rightarrow L(x)$.

Let $x_0, x_1, \ldots, x_n$ be the nonterminals of $G$, and $x_i \rightarrow t_{i,1} | \cdots | t_{i,k_i}$ the productions.

\[
\begin{align*}
\vdash \varphi(t_{0,1}), \ldots, \varphi(t_{0,k_n}) \\
\vdash \exists x \varphi(x) \\
\vdash \varphi(t_{n,1}), \ldots, \varphi(t_{n,k_0}) \\
\vdash \exists x \varphi(x) \\
\vdash \varphi(x_n) \vdash \exists x \varphi(x) \\
\exists x \varphi(x) \vdash \exists x \varphi(x) \\
\vdash \exists x \varphi(x)
\end{align*}
\]
Corresponding complexity results for simple proofs

Problem (H-membership)
Let $\pi$ be a simple proof, and $f$ a formula. Is there a $ne$-$NF$ $\pi \xrightarrow{ne} \pi^*$ such that $f \in H(\pi^*)$?

$\Rightarrow$ NP-complete

Problem (H-containment)
Let $\pi_1, \pi_2$ be simple proofs. Are there $ne$-$NFs$ $\pi_i \xrightarrow{ne} \pi_i^*$ such that $H(\pi_1^*) \subseteq H(\pi_2^*)$?

$\Rightarrow$ $\Pi_2^P$-complete
Problem (H-disjointness)
Let $\pi_1, \pi_2$ be simple proofs. Are there $\neq\nsim$-NFs $\pi_i \neq\nsim \pi_i^*$ such that $H(\pi_1^*) \cap H(\pi_2^*) = \emptyset$?

$\Rightarrow$ coNP-complete

Problem (H-equivalence)
Let $\pi_1, \pi_2$ be simple proofs. Are there $\neq\nsim$-NFs $\pi_i \neq\nsim \pi_i^*$, such that $H(\pi_1^*) = H(\pi_2^*)$?

$\Rightarrow$ $\Pi_2^P$-complete
Conclusion

• We can analyze cut-reduction using tree grammars.

Future work:

• Given a set of terms $T$ and $n \geq 0$, is there a TRATG $G$ such that $T \subseteq L(G)$ with at most $n$ productions?
  • NP-complete if $G$ has two nonterminals, otherwise unknown.